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Eikonal groups of photon and phonon propagators in one-dimensional structures

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Abstract. A general tensor eikonal method is proposed for solving wave equations of the optics and acoustics of stratified anisotropic media in the case of non-commutation of the tensor eikonals. Such an approach allows one to operate with electromagnetic and acoustic fields in inhomogeneous anisotropic media without any division of them into partial waves and not referring to a particular coordinate systems. The tensor eikonals which are determined by the polarization of waves are involved in the evolution operators (propagators) and are expressed in terms of the normal refraction tensors. These tensors are non-commutative in the general case and disentangling the evolution operators is necessary, including ones for waves in isotropic media. Such disentangling is performed with the use of known standard operator procedures. An example of the calculation of the photon propagator is given for a stratified medium with the inhomogeneity profile $\varepsilon(z) = a + b/z^2$.

1. Introduction

In recent years increasing attention has been paid to the investigations of topological phases [1], photonic and phononic crystals [2], spatial solitons [3], the topology of light traps and mirages [4]. In numerous works on these problems, the important role of the angular momenta of photons and phonons has been revealed, and the relationships between ray and wave notions used in geometrical optics have been pointed out [5]. There is a vast literature on the different approaches in geometrical optics. In one form or another, geometro-optical constructions were taken up by Hamilton, Bruns, Sommerfeld, Debye, Runge, de Broglie, Keller and others [6]. The question is one of asymptotic constructions on the basis of wave theory. Despite the great variety of different methods in this field there has been very little work which has allowed a comparatively simple evaluation of the role of angular momenta in the ray representation. For instance, in the book by Marcuse [7] it is asserted that photon spin is neglected in the ideas of geometrical optics. This pertains naturally to constructions of scalar geometrical optics [6, 8]. Earlier in [9] a tensor generalization of scalar eikonals in optics and acoustics of stratified media was given. In these works, however, non-commutation of the tensor eikonal was not taken into account, and the evolution operators which characterize the field variation in three-dimensional space were presented as ordinary exponentials. Meanwhile such non-commutation takes place in the overwhelming majority of practically important cases and is worthy of separate consideration. In this paper we expound a calculus of the generalized tensor eikonals for oblique incidence of photons or phonons on a stratified medium when non-commutation occurs. In so doing, we rely on the mathematical work of Baker [10] which have, up to now, stimulated study on Lie groups [11]. We carry out a consideration in terms of the operators of the spatial evolution of the optical and acoustic fields with the use of an

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opto-acoustic analogy [12]. Under certain conditions these operators form continuous Lie groups, the generators of which are photon and phonon eikonals. In quantum electrodynamics (QED) there are many forms of representation of propagators for different particles. In the literature these propagators are called Green operators, Cauchy operators and shift operators [13]. In this paper we try to emphasize the group nature of the operators indicated and for understandable reasons call the corresponding groups *eikonal groups*. We are of one mind with Brodsky and Drell concerning the charming simplicity of Maxwell's equations formulated using field intensities (not potentials) [13] not only for the purposes of QED but for classical and semiclassical constructions [12]. Maxwell's equations themselves point to the possibility of operator generalization of the main wave characteristics such as refractive index, frequency, impedance and so on. Eikonal are also among such characteristics.

This paper consists of an introduction, a main part (sections 2–4) and a conclusion. In section 2 we consider the general case of the oblique incidence of electromagnetic or acoustic waves on stratified anisotropic media and describe a procedure for tensor eikonal approximations. Such a procedure generalizes the scalar eikonal approximations and consistently takes into account not only wave polarizations but electromagnetic and acoustic fields as a whole without their division into partial waves. It demonstrates the main difference of our approach from others where partial waves are considered separately. In section 3 we apply the results obtained to isotropic stratified media. We show that even in this case the eikonal tensors and the normal refraction tensors associated with them do not commute when taken at different layers. We propose a procedure for calculation of the electromagnetic fields in such media using operator methods [15]. There is comparatively small number of inhomogeneity profiles for stratified media leading to exact solutions of the wave equations [14]. In section 4 we take, as an example, a dielectric permittivity profile $\varepsilon(z) = a + b/z^2$ and compare the exact solution of the original wave equation with its operator eikonal solution at oblique incidence. It is shown that at certain angles of incidence the tensor geometro-optical solutions coincide with the exact ones already in the zeroth-order approximation.

2. Tensor eikonal equations for a stratified general anisotropic medium

Let us consider a monochromatic wave (electromagnetic or elastic) with time dependence $\exp(-\mathrm{i}\omega t)$ propagating in a linear inhomogeneous anisotropic medium. In this case the field distribution $\boldsymbol{F}(\boldsymbol{r})$ in the medium as a function of the radius vector $\boldsymbol{r}$ of the observation point, can be described by the following generalized Helmholtz equation:

$$\frac{\partial}{\partial r_b}\left(\sigma_{abcd}\frac{\partial}{\partial r_c}F_d\right) - g^2\nu_{ad}F_d = 0. \tag{1}$$

In (1) depending on whether an electromagnetic or elastic wave is considered, the vector $\boldsymbol{F}(\boldsymbol{r})$ characterizes either a magnetic field intensity $\boldsymbol{H}(\boldsymbol{r})$ or a displacement $\boldsymbol{u}(\boldsymbol{r})$. For electromagnetic waves $\sigma_{abcd} = e_{abf}\varepsilon^{-1}_{fg}e_{gcd}$, $g = k = \omega/c$, $\nu_{ab} = \mu_{ab}$ with the constitutive equations assumed to be $\boldsymbol{D} = \varepsilon(\boldsymbol{r})\boldsymbol{E}$, $\boldsymbol{B} = \mu(\boldsymbol{r})\boldsymbol{H}$ (ε is the dielectric permittivity tensor, μ is the magnetic permeability tensor, c is the speed of light in vacuum, e_{abc} is the Levi-Civita pseudotensor) and for elastic waves $\sigma_{abcd} = c_{abcd}$ are elastic stiffnesses, $g = \omega$, $\nu_{ab} = -\rho\delta_{ab}$, ρ is the density of the medium and δ_{ab} is the Kronecker delta. Summation is over repeated indices.

If the medium is stratified in the z-direction then the wave stimulated under oblique incidence is described with the equation

$$\boldsymbol{F}(\boldsymbol{r}) = \boldsymbol{F}(z)\exp(\mathrm{i}g\boldsymbol{b}\cdot\boldsymbol{r}). \tag{2}$$

In formula (2) the field amplitude $\boldsymbol{F}(\boldsymbol{r})$ depends on both the longitudinal coordinate $z = \boldsymbol{q} \cdot \boldsymbol{r}$ along the stratification direction ($\boldsymbol{q}$ is an unit normal vector co-directed with the z-axis) and transverse coordinate x along the $\boldsymbol{b}$-direction. The vector $\boldsymbol{b}$ is perpendicular to the unit vector $\boldsymbol{q}$ and is parallel to the stratification planes. It is highly convenient to use the vector $\boldsymbol{b}$ in solving a series of problems of optics and acoustics with oblique incidence of waves. The geometrical meaning of $\boldsymbol{b}$ is revealed to its full extent on consideration of the refraction and reflection of waves on the plane interfaces [16]. $\boldsymbol{b}^2$ is determined by the incidence angle. At normal incidence $\boldsymbol{b} = 0$ ($\boldsymbol{b}^2 = 0$) and therefore the field amplitude $\boldsymbol{F}(\boldsymbol{r})$ depends on the z coordinate only.

For the wave of type (2) equation (1) can be rewritten in index-free notation as the following:

$$-g^2(B+\nu)\boldsymbol{F}(z) + \mathrm{i}g\left[\frac{\mathrm{d}C}{\mathrm{d}z}\boldsymbol{F}(z) + (C+S)\frac{\mathrm{d}\boldsymbol{F}(z)}{\mathrm{d}z}\right] + \frac{\mathrm{d}Q}{\mathrm{d}z}\frac{\mathrm{d}\boldsymbol{F}(z)}{\mathrm{d}z} + Q\frac{\mathrm{d}^2\boldsymbol{F}(z)}{\mathrm{d}z^2} = 0. \tag{3}$$

In (3) we have introduced the tensors B, Q, C and S,

$$B_{ab} = b_c\sigma_{acdb}b_d \qquad Q_{ab} = q_c\sigma_{acdb}q_d \qquad C_{ab} = q_c\sigma_{acdb}b_d \qquad S_{ab} = b_c\sigma_{acdb}q_d \tag{4}$$

and, for example, $(\mathrm{d}C/\mathrm{d}z)\boldsymbol{F}(z)$ denotes in index notation $(\mathrm{d}C_{ab}/\mathrm{d}z)F_b(z)$.

For a homogeneous anisotropic medium the solution of equation (3) can be represented in evolution form [17]

$$\boldsymbol{F}(z) = \exp(\mathrm{i}gNz)\boldsymbol{F}(z_0) \tag{5}$$

where N is the second-rank tensor called the *normal refraction tensor* and $\boldsymbol{F}(z_0)$ is the amplitude in the reference point z_0 which is assumed to be given.

It is known that the introduction of eikonal of waves in the case of inhomogeneous isotropic media is based on the solution with a scalar phase function in the exponent. Making the transition to inhomogeneous anisotropic media it is natural to use the solution (5) with the tensor phases. If for typical wavelengths the properties of the medium change to a small extent, then the corresponding solution of the wave equation should have little difference in comparison with (5). Therefore, approximate solutions will have a form analogous to (5), but with variable tensors N and vectors of amplitudes. Relying on (5) we shall search for the solution of equation (3) in the form

$$\boldsymbol{F}(z) \cong \Omega_{z_0}^{z}[\mathrm{i}gN(z)]\boldsymbol{A}(z) = \widehat{\int_{z_0}^{z}}[1 + \mathrm{i}gN(z')\,\mathrm{d}z']\boldsymbol{A}(z) \tag{6}$$

where $\boldsymbol{A}(z)$ is a slowly varying amplitude. The symbol $\Omega_{z_0}^{z}[\mathrm{i}gN(z)]$ denotes a matrizant (multiplicative integral, integral exponent) [18] and generalizes a tensor exponential. The specific nature of the multiplicative integral is to a great extent connected with the non-commutation of the operators $N(z)$. If operators $N(z')$ and $N(z'')$ are permutable at two arbitrary points z' and z'': $N(z')N(z'') = N(z'')N(z')$; $z', z'' \in [z_0, z]$, then the multiplicative integral reduces to the operator $\exp[\mathrm{i}g\int_{z_0}^{z} N(z)\,\mathrm{d}z]$. However, in the general case of non-commutation we introduce the notation

$$\Psi(z) = \int_{z_0}^{z} N_z(z)\,\mathrm{d}z. \tag{7}$$

The lower index of the operator N denotes ordering of operators. If $\Psi(z)$ (7) is substituted in an exponent and this exponent is expanded into a series then in each term of the series

$N_z N_{z'} = N(z)N(z')$ when $z > z'$ and $N_z N_{z'} = N(z')N(z)$ when $z' > z$. The operator provided with the larger index acts later [19]. Then we can rewrite (6) as follows:

$$\boldsymbol{F}(z) = \exp[\mathrm{i}g\Psi(z)]\boldsymbol{A}(z). \tag{8}$$

We shall call the tensor function $\Psi(z)$ the *eikonal tensor* for the field $\boldsymbol{F}(z)$.

Now let us expand the amplitude $\boldsymbol{A}(z)$ into power series of i/g (a Debye expansion), assuming that the series is convergent,

$$\boldsymbol{A}(z) = \boldsymbol{A}_0(z) + \frac{\mathrm{i}}{g}\boldsymbol{A}_1(z) + \left(\frac{\mathrm{i}}{g}\right)^2 \boldsymbol{A}_2(z) \ldots. \tag{9}$$

Substituting (8) and (9) in (3) and comparing the coefficients at the same degrees of i/g, in the zeroth-order approximation we obtain the equation

$$[QN^2 + (S + C)N + B + \nu]\boldsymbol{F}_0(z) = 0. \tag{10}$$

Note that the equality (10) should hold for any z. Therefore, from (10) the tensor equation for finding the eikonal takes the form

$$QN^2 + (S + C)N + B + \nu = 0. \tag{11}$$

For homogeneous media the tensor N in (11) is constant and it coincides with the normal refraction tensor [17]. The general expressions (10) and (11) are obtained with the supposition that tensors N taken at the different medium layers do not commute.

For higher-order approximations we have recurrence relations:

$$\begin{aligned}
&[QN^2 + (S + C)N + B + \nu]\Omega_{z_0}^z[\mathrm{i}gN(z)]\boldsymbol{A}_{j+1}(z) - \frac{\mathrm{d}(QN + C)}{\mathrm{d}z}\Omega_{z_0}^z[\mathrm{i}gN(z)]\boldsymbol{A}_j(z)\\
&\quad -(2QN + S + C)\Omega_{z_0}^z[\mathrm{i}gN(z)]\frac{\mathrm{d}\boldsymbol{A}_j(z)}{\mathrm{d}z} + Q\Omega_{z_0}^z[\mathrm{i}gN(z)]\frac{\mathrm{d}^2\boldsymbol{A}_{j-1}(z)}{\mathrm{d}z^2}\\
&\quad +\frac{\mathrm{d}Q}{\mathrm{d}z}\Omega_{z_0}^z[\mathrm{i}gN(z)]\frac{\mathrm{d}\boldsymbol{A}_{j-1}(z)}{\mathrm{d}z} = 0
\end{aligned} \tag{12}$$

where $j = 0, 1, 2, \ldots$; $\boldsymbol{A}_j = 0$ at $j < 0$.

The solution of (12) can be expressed in terms of multiplicative integrals

$$\boldsymbol{A}_j(z) = \Omega_{z_0}^z[G(z)]\boldsymbol{A}_j(z_0) + \int_{z_0}^z K(z, z')H(z')\boldsymbol{A}_{j-1}(z')\,\mathrm{d}z' \tag{13}$$

where $N(z)$ is supposed to be found from (11),

$$G(z) = -\left\{\Omega_{z_0}^z[\mathrm{i}gN(z)]\right\}^{-1}(2QN + S + C)^{-1}\frac{\mathrm{d}}{\mathrm{d}z}(QN + C)\Omega_{z_0}^z[\mathrm{i}gN(z)]$$

and

$$H(z) = \left\{\Omega_{z_0}^z[\mathrm{i}gN(z)]\right\}^{-1}(2QN + S + C)^{-1}\left\{\frac{\mathrm{d}Q}{\mathrm{d}z}\Omega_{z_0}^z[\mathrm{i}gN(z)]\frac{\mathrm{d}}{\mathrm{d}z} + Q\Omega_{z_0}^z[\mathrm{i}gN(z)]\frac{\mathrm{d}^2}{\mathrm{d}z^2}\right\}$$

$$K(z, z') = \Omega_{z_0}^z[G(z)]\Omega_{z'}^{z_0}[G(z')] \qquad \left\{\Omega_{z_0}^z[\mathrm{i}gN(z)]\right\}^{-1} = \exp\left[\mathrm{i}g\int_z^{z_0} N_z(z)\,\mathrm{d}z\right].$$

The relations obtained above enable consideration of the spatial evolution of the amplitude $\boldsymbol{F}(z)$ of the magnetic field or displacement in the tensor geometro-optical approximation. For electromagnetic waves the electric field $\boldsymbol{E}(z)$ can be restored from the known $\boldsymbol{H}(z)$ using Maxwell's equations.

Below we find the tensor eikonal for electromagnetic waves at oblique incidence to an isotropic stratified medium and disentangle the evolution operator in the zeroth-order approximation.

3. Non-Abelian eikonal algebras for stratified isotropic media

The general expressions (6), (10) and (11) demonstrate that the eikonal is a tensor value even in isotropic media. We apply these expressions in the zeroth-order approximation to the case of oblique incidence of an electromagnetic wave on a stratified isotropic medium. It appears that even in this case the eikonals taken at different points in the medium will form a field of non-commuting tensors and in the geometro-optical approximation it is necessary to disentangle the evolution operator.

Let an isotropic stratified medium be described by the scalar permittivity $\varepsilon = \varepsilon(z)$ and the scalar permeability $\mu = \mu(z)$. The tensor σ_{abcd} included in the Helmholtz equation (1) takes the form

$$\sigma_{abcd} = \frac{1}{\varepsilon} e_{abi} e_{cdi} = \frac{1}{\varepsilon}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}). \tag{14}$$

Substituting expression (14) in formulae (4), for the tensors B, Q, S, C we find in index-free notation [16]

$$\begin{aligned} B &= \frac{1}{\varepsilon}(\boldsymbol{b}\otimes\boldsymbol{b} - \boldsymbol{b}^2) \qquad & Q &= \frac{1}{\varepsilon}(\boldsymbol{q}\otimes\boldsymbol{q} - 1) = -\frac{1}{\varepsilon} I \\ S &= \frac{1}{\varepsilon}\boldsymbol{q}\otimes\boldsymbol{b} & C &= \frac{1}{\varepsilon}\boldsymbol{b}\otimes\boldsymbol{q} \end{aligned} \tag{15}$$

where the vectors $\boldsymbol{b}$ and $\boldsymbol{q}$ introduced in section 2 are used (see formulae (2) and (4)), $\boldsymbol{b}\otimes\boldsymbol{b}$, $\boldsymbol{q}\otimes\boldsymbol{q}$, $\boldsymbol{q}\otimes\boldsymbol{b}$, $\boldsymbol{b}\otimes\boldsymbol{q}$ are dyads, $I = 1-\boldsymbol{q}\otimes\boldsymbol{q}$ is the projective operator on a plane which is perpendicular to the vector $\boldsymbol{q}$. Notes of the type $\boldsymbol{b}\otimes\boldsymbol{b} - \boldsymbol{b}^2$ should be understood as $\boldsymbol{b}\otimes\boldsymbol{b} - \boldsymbol{b}^2 1$. Having substituted (15) into (11), we obtain the following square equation for the tensor $N = N(z)$:

$$IN^2 - (\boldsymbol{b}\otimes\boldsymbol{q} + \boldsymbol{q}\otimes\boldsymbol{b})N - \xi - \boldsymbol{b}\otimes\boldsymbol{b} = 0 \tag{16}$$

where the designation $\xi(z) = \varepsilon(z)\mu(z) - \boldsymbol{b}^2$ is introduced. We search for solutions of this equation in the form

$$N = \alpha\boldsymbol{b}\otimes\boldsymbol{b} + \beta\boldsymbol{b}\otimes\boldsymbol{q} + \gamma\boldsymbol{q}\otimes\boldsymbol{b} + \eta\boldsymbol{q}\otimes\boldsymbol{q} + \lambda\boldsymbol{a}\otimes\boldsymbol{a} \tag{17}$$

where $\boldsymbol{a} = [\boldsymbol{bq}]$ is a vector product of vectors $\boldsymbol{b}$ and $\boldsymbol{q}$. Substitution of (17) into (16) with subsequent equating factors at identical dyads results in a system of algebraic equations for the unknown variables α, β, γ, η, λ. Its solution gives

$$N(z) = \pm\frac{1}{\boldsymbol{b}^2}\sqrt{\xi(z)}\boldsymbol{a}\otimes\boldsymbol{a} - \frac{1}{\boldsymbol{b}^2}\xi(z)\boldsymbol{b}\otimes\boldsymbol{q} - \boldsymbol{q}\otimes\boldsymbol{b}. \tag{18}$$

It is easy to show that in the general case the tensors $N(z')$ and $N(z'')$ taken at the not coinciding arguments do not commute (an exception is the case of the homogeneous medium when ε, μ and ξ do not depend on z). It means that the tensor eikonal $\Psi(z)$ (7) cannot be presented in the form of an ordinary integral $\int_{z_0}^{z} N(z')\,\mathrm{d}z'$, and the field solutions in the zeroth-order approximation are described by the formula

$$\boldsymbol{H}_0(z) = \Omega_{z_0}^{z}[\mathrm{i}kN(z)]\boldsymbol{H}(z_0) \tag{19}$$

where $\Omega_{z_0}^{z}[\mathrm{i}kN(z)] = \widehat{\int_{z_0}^{z}}[1 + \mathrm{i}kN(z')\,\mathrm{d}z']$ is the evolution operator and $\boldsymbol{H}(z_0)$ is the vector of the magnetic field strength, which is assumed to be given at a reference point with coordinate z_0.

We shall calculate $\Omega_{z_0}^{z}[\mathrm{i}kN(z)]$ following the standard Wei–Norman procedure [15]. The multiplicative integral $\Omega_{z_0}^{z}[\mathrm{i}kN(z)]$ satisfies the operator differential equation

$$\frac{\mathrm{d}\Omega_{z_0}^{z}}{\mathrm{d}z}[\Omega_{z_0}^{z}]^{-1} = \mathrm{i}kN(z) \tag{20}$$

under the condition $\Omega_{z_0}^{z_0} = 1$. Equation (20) follows from the fact that $\Omega_{z_0}^{z}[\mathrm{i}kN(z)]$ can be represented in the form $(1 + \mathrm{i}kN(z_n)\Delta z)(1 + \mathrm{i}kN(z_{n-1})\Delta z)\ldots(1 + \mathrm{i}kN(z_0)\Delta z)$ at $\Delta z \to 0$ where $z_{i+1} = z_i + \Delta z$, $z_n = z$. So if we differentiate $\Omega_{z_0}^{z}$ with respect z this is equivalent to multiplication of $\Omega_{z_0}^{z}$ by $\mathrm{i}kN(z)$ from the left: $\mathrm{d}\Omega_{z_0}^{z}/\mathrm{d}z = \mathrm{i}kN(z)\Omega_{z_0}^{z}$ [18]. Now we introduce the operators

$$L_1 = \boldsymbol{b} \otimes \boldsymbol{q} \qquad L_2 = \boldsymbol{q} \otimes \boldsymbol{b} \qquad L_3 = \boldsymbol{b} \otimes \boldsymbol{b} - \boldsymbol{b}^2\boldsymbol{q} \otimes \boldsymbol{q} \qquad L_4 = \boldsymbol{a} \otimes \boldsymbol{a}. \tag{21}$$

Then we can rewrite the tensor $N(z)$ (18) in the form

$$N(z) = -\frac{1}{\boldsymbol{b}^2}\xi(z)L_1 - L_2 \pm \frac{1}{\boldsymbol{b}^2}\sqrt{\xi(z)}L_4. \tag{22}$$

The operators L_1, L_2, L_3, L_4 (21) and their linear combinations form a non-Abelian Lie algebra with the commutation rules

$$\begin{aligned} &[L_1, L_2] = L_3 \qquad [L_1, L_3] = -2\boldsymbol{b}^2 L_1 \qquad [L_2, L_3] = 2\boldsymbol{b}^2 L_2 \\ &[L_1, L_4] = [L_2, L_4] = [L_3, L_4] = 0. \end{aligned} \tag{23}$$

We shall present $\Omega_{z_0}^{z}[\mathrm{i}kN(z)]$ as an expansion

$$\Omega_{z_0}^{z}[\mathrm{i}kN(z)] = \exp[\mathrm{i}g_1(z)L_1]\exp[\mathrm{i}g_2(z)L_2]\exp[g_3(z)L_3]\exp[\mathrm{i}g_4(z)L_4] \tag{24}$$

where $g_1(z)$, $g_2(z)$, $g_3(z)$ and $g_4(z)$ are some functions of the variable z, $k = \omega/c$. Then the left-hand part of equation (20) using the commutation rules (23) and the Baker–Hausdorff formula [10]

$$\mathrm{e}^X Y \mathrm{e}^{-X} = Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] + \frac{1}{3!}[X, [X, [X, Y]]] + \cdots$$

can be written as the following:

$$\begin{aligned} \frac{\mathrm{d}\Omega_{z_0}^{z}}{\mathrm{d}z}[\Omega_{z_0}^{z}]^{-1} &= \mathrm{i}\frac{\mathrm{d}g_1}{\mathrm{d}z}L_1 + \mathrm{i}\frac{\mathrm{d}g_4}{\mathrm{d}z}L_4 + \mathrm{i}\frac{\mathrm{d}g_2}{\mathrm{d}z}(\boldsymbol{b}^2 g_1^2 L_1 + L_2 + \mathrm{i}g_1 L_3) \\ &\quad + \frac{\mathrm{d}g_3}{\mathrm{d}z}[2\mathrm{i}\boldsymbol{b}^2 g_1(\boldsymbol{b}^2 g_1 g_2 - 1)L_1 + 2\mathrm{i}\boldsymbol{b}^2 g_2 L_2 + (1 - 2\boldsymbol{b}^2 g_1 g_2)L_3]. \end{aligned} \tag{25}$$

We shall substitute (22) and (25) into (20) and equate factors for identical operators. Then we obtain the system of nonlinear differential equations of first order,

$$\begin{aligned} &\frac{\mathrm{d}g_1}{\mathrm{d}z} = -\frac{k}{\boldsymbol{b}^2}\xi(z) - k\boldsymbol{b}^2 g_1^2 \qquad &&\frac{\mathrm{d}g_3}{\mathrm{d}z} = -kg_1 \\ &\frac{\mathrm{d}g_2}{\mathrm{d}z} = k(2\boldsymbol{b}^2 g_1 g_2 - 1) \qquad &&\frac{\mathrm{d}g_4}{\mathrm{d}z} = \pm\frac{k}{\boldsymbol{b}^2}\sqrt{\xi(z)} \end{aligned} \tag{26}$$

under the initial conditions $g_1(z_0) = g_2(z_0) = g_3(z_0) = g_4(z_0) = 0$. The first equation of the system is the Riccati equation concerning the function $g_1(z)$. It can be solved in quadrature only for some special functions $\xi = \xi(z)$ (see section 4). If an explicit form $g_1 = g_1(z)$ of the solution of this equation is found, the solutions $g_3 = g_3(z)$ and $g_2 = g_2(z)$ are under the formulae

$$g_3(z) = -k\int_{z_0}^{z} g_1(z')\,\mathrm{d}z' \qquad g_2(z) = -k\int_{z_0}^{z} \exp[-2\boldsymbol{b}^2(g_3(z) - g_3(z'))]\,\mathrm{d}z'. \tag{27}$$

Solutions of the fourth equation of the system (26) are

$$g_4(z) = \pm\frac{k}{\boldsymbol{b}^2}\int_{z_0}^{z}\sqrt{\xi(z')}\,\mathrm{d}z' = \pm\frac{k}{\boldsymbol{b}^2}\int_{z_0}^{z}\sqrt{\varepsilon(z')\mu(z') - \boldsymbol{b}^2}\,\mathrm{d}z'. \tag{28}$$

Now expanding the exponents in (24) under ordinary rules and taking into account that $(L_1)^2 = (L_2)^2 = 0$, $(L_3)^2 = \boldsymbol{b}^2(\boldsymbol{b}\otimes\boldsymbol{b} + \boldsymbol{b}^2\boldsymbol{q}\otimes\boldsymbol{q}) = (\boldsymbol{b}^2)^2 I$, $(L_4)^2 = \boldsymbol{b}^2\boldsymbol{a}\otimes\boldsymbol{a} = \boldsymbol{b}^2 L_4$ we arrive at the following formula for the operator $\Omega_{z_0}^{z}[\mathrm{i}kN(z)]$ as a linear combination of dyads:

$$\begin{aligned}\Omega_{z_0}^{z}[\mathrm{i}kN(z)] = {}& \frac{1}{\boldsymbol{b}^2}\exp(g_3\boldsymbol{b}^2)(1 - \boldsymbol{b}^2 g_1 g_2)\boldsymbol{b}\otimes\boldsymbol{b} \\ & + \mathrm{i}g_1\exp(-g_3\boldsymbol{b}^2)\boldsymbol{b}\otimes\boldsymbol{q} + \mathrm{i}g_2\exp(g_3\boldsymbol{b}^2)\boldsymbol{q}\otimes\boldsymbol{b} \\ & + \exp(-g_3\boldsymbol{b}^2)\boldsymbol{q}\otimes\boldsymbol{q} + \frac{1}{\boldsymbol{b}^2}\exp(\mathrm{i}g_4\boldsymbol{b}^2)\boldsymbol{a}\otimes\boldsymbol{a}.\end{aligned} \tag{29}$$

Thus the expression (29) together with the first equation of the system (26) and formulae (27) and (28) completely determines the evolution operator $\Omega_{z_0}^{z}[\mathrm{i}kN(z)]$ involved in (19). The relations obtained above can be applied for calculations of electromagnetic fields in the tensor geometro-optical approximation if the functional dependences $\varepsilon = \varepsilon(z)$ and $\mu = \mu(z)$ are given in an explicit form.

4. Example of tensor eikonal approximations and comparison with the exact solution

Now we apply the general results of sections 2 and 3 for electromagnetic waves propagating in an inhomogeneous isotropic medium with the profile

$$\varepsilon(z) = a + \frac{b}{z^2} \qquad \mu(z) = 1 \tag{30}$$

where a and b are some constant quantities which characterize the medium. Let suppose that the parameter $\boldsymbol{b}^2$ associated with oblique incidence of the wave coincides with a (i.e. the angle of incidence is fixed and determined by the parameter a). We make such an assumption to obtain a simpler solution of equations (26) as far as possible and then compare it with an exact solution of Maxwell's equations. Really, in this case the quantity $\xi(z)$ becomes equal to b/z^2 and the first equation of system (26) (the Riccati equation) allows an analytical solution. Thereby other functions $g_2(z)$, $g_3(z)$, $g_4(z)$ and the evolution operator in the zeroth-order approximation (29) can also be represented in analytical form. It turns out that for the waves polarized in the plane of incidence (in the plane passing through vectors $\boldsymbol{b}$ and $\boldsymbol{q}$) already in a zeroth-order approximation will coincide with the exact solution of Maxwell's equations for the profile (30).

The Riccati equation for $g_1(z)$

$$\frac{\mathrm{d}g_1}{\mathrm{d}z} = -\frac{kb}{az^2} - kag_1^2 \tag{31}$$

has the partial solution $g_1(z) = d/z$. Substituting it into (31) we find that

$$d = \frac{1}{2ka}(1 \pm \sqrt{1 - 4k^2 b}). \tag{32}$$

Proceeding with the partial solution obtained we construct the general solution in the form

$$g_1(z) = \frac{d}{z} + \frac{1}{f(z)} \tag{33}$$

where $f(z)$ is some function to be determined. Substituting (33) in (31) and solving equation (31) for $f(z)$, using (32) we arrive at an expression for $g_1(z)$

$$g_1(z) = \frac{1}{z}\left[d + \left(\frac{ka}{1-2\gamma} + Cz^{2\gamma-1}\right)^{-1}\right] \tag{34}$$

where the designation $\gamma = kad$ is introduced and C is a constant of integration. Under the initial condition $g_1(z_0) = 0$ the final expression for the function $g_1(z)$ is

$$g_1(z) = \frac{d(1-\gamma)}{\gamma z}\left[1 + \frac{(1-2\gamma)(z/z_0)^{2\gamma-1}}{\gamma-(1-\gamma)(z/z_0)^{2\gamma-1}}\right]. \tag{35}$$

Using formulae (27) and (28) we find in the succession functions $g_3(z)$, $g_2(z)$ and $g_4(z)$ under the initial conditions $g_3(z_0) = g_2(z_0) = g_4(z_0) = 0$

$$\begin{aligned} g_3(z) &= -\frac{kd}{\gamma}\ln\left\{\frac{(z/z_0)^{1-\gamma}}{2\gamma-1}\left[\gamma-(1-\gamma)(z/z_0)^{2\gamma-1}\right]\right\} \\ g_2(z) &= \frac{kz_0(z/z_0)^{2(1-\gamma)}}{(2\gamma-1)^2}\left[1-(z/z_0)^{2\gamma-1}\right]\left[\gamma-(1-\gamma)(z/z_0)^{2\gamma-1}\right] \\ g_4(z) &= \pm\frac{k\sqrt{b}}{a}\ln\frac{z}{z_0}. \end{aligned} \tag{36}$$

Then substituting the functions $g_i(z)$, $i = 1, \ldots, 4$ (35) and (36) in (29) after uncomplicated transformations we obtain the evolution operator $\Omega^z_{z_0}[\mathrm{i}kN(z)]$ in the zeroth-order geometro-optical approximation,

$$\begin{aligned} \Omega^z_{z_0}[\mathrm{i}kN(z)] = {} & \frac{1}{(2\gamma-1)a}\left[(\gamma-1)(z/z_0)^{-\gamma} + \gamma(z/z_0)^{\gamma-1}\right]\boldsymbol{b}\otimes\boldsymbol{b} \\ & + \frac{\mathrm{i}kb}{(2\gamma-1)z_0 a}\left[(z/z_0)^{-\gamma} - (z/z_0)^{\gamma-1}\right]\boldsymbol{b}\otimes\boldsymbol{q} \\ & + \frac{\gamma(\gamma-1)z_0}{\mathrm{i}kb(2\gamma-1)}\left[(z/z_0)^{-(\gamma-1)} - (z/z_0)^{\gamma}\right]\boldsymbol{q}\otimes\boldsymbol{b} \\ & + \frac{1}{2\gamma-1}\left[\gamma(z/z_0)^{-(\gamma-1)} + (\gamma-1)(z/z_0)^{\gamma}\right]\boldsymbol{q}\otimes\boldsymbol{q} \\ & + \frac{1}{a}\{\cos[k\sqrt{b}\ln(z/z_0)] \pm \mathrm{i}\sin[k\sqrt{b}\ln(z/z_0)]\}\boldsymbol{a}\otimes\boldsymbol{a}. \end{aligned} \tag{37}$$

Note that value of $\Omega^z_{z_0}[\mathrm{i}kN(z)]$ does not depend on whether the upper or lower sign is chosen in formula (32) for the quantity d which is involved in (37) through the parameter $\gamma = kad$. In reality, a change of sign in (32) from negative to positive (or vice versa) leads to the replacements $\gamma \longrightarrow -(\gamma-1)$, $2\gamma-1 \longrightarrow -(2\gamma-1)$ in (37) which do not change the value of $\Omega^z_{z_0}[\mathrm{i}kN(z)]$. The evolution operator is expressed in terms of power functions of z/z_0, exponents of a power being complex quantities with the condition $k^2 b \gg 1$. This condition determines the applicability of the geometro-optical approximation in the case under consideration.

Now we turn to an exact solution for the profile $\varepsilon(z) = a + (b/z^2)$. The dependence of the tangential components of the magnetic and electric field for a stratified anisotropic dielectric medium is described by a matrix system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{pmatrix}\boldsymbol{H}_\tau \\ [\boldsymbol{qE}]\end{pmatrix} = \mathrm{i}k\mathbb{M}(z)\begin{pmatrix}\boldsymbol{H}_\tau \\ [\boldsymbol{qE}]\end{pmatrix} \tag{38}$$

with the matrix $\mathbb{M}(z)$ having a block structure [17]

$$\mathbb{M}(z) = \begin{pmatrix} \boldsymbol{q}^{\times}\varepsilon\boldsymbol{q}\otimes\boldsymbol{a}/\boldsymbol{q}\varepsilon\boldsymbol{q} & -\boldsymbol{b}\otimes\boldsymbol{b}+I\bar{\tilde{\varepsilon}}I/\boldsymbol{q}\varepsilon\boldsymbol{q} \\ I-\boldsymbol{a}\otimes\boldsymbol{a}/\boldsymbol{q}\varepsilon\boldsymbol{q} & -\boldsymbol{a}\otimes\boldsymbol{q}\varepsilon\boldsymbol{q}^{\times}/\boldsymbol{q}\varepsilon\boldsymbol{q} \end{pmatrix} \tag{39}$$

where $\boldsymbol{H}_\tau = I\boldsymbol{H}$, $\boldsymbol{q}^{\times}$ is the tensor dual to vector $\boldsymbol{q}$, a tilde denotes a transposed tensor and a bar denotes an adjugate tensor [16]. Complete three-dimensional vectors $\boldsymbol{H}$ and $\boldsymbol{E}$ are restored from their tangential components according to the formula

$$\begin{pmatrix} \boldsymbol{H} \\ \boldsymbol{E} \end{pmatrix} = \mathbb{V}(z)\begin{pmatrix} \boldsymbol{H}_\tau \\ [\boldsymbol{q}\boldsymbol{E}] \end{pmatrix} \tag{40}$$

where

$$\mathbb{V}(z) = \begin{pmatrix} I & -\boldsymbol{q}\otimes\boldsymbol{a} \\ \boldsymbol{q}\otimes\boldsymbol{a}/\boldsymbol{q}\varepsilon\boldsymbol{q} & -\boldsymbol{q}^{\times}+\boldsymbol{q}\otimes\boldsymbol{q}\varepsilon\boldsymbol{q}^{\times}/\boldsymbol{q}\varepsilon\boldsymbol{q} \end{pmatrix}. \tag{41}$$

In (39) and (41) $\varepsilon = \varepsilon(z)$ is the dielectric permittivity tensor of the anisotropic medium. For the isotropic medium $\varepsilon(z)$ is considered as a scalar quantity and the matrices $\mathbb{M}(z)$ (39) and $\mathbb{V}(z)$ (41) take the form

$$\mathbb{M}(z) = \begin{pmatrix} 0 & \varepsilon I-\boldsymbol{b}\otimes\boldsymbol{b} \\ I-(1/\varepsilon)\boldsymbol{a}\otimes\boldsymbol{a} & 0 \end{pmatrix} \qquad \mathbb{V}(z) = \begin{pmatrix} I & -\boldsymbol{q}\otimes\boldsymbol{a} \\ (1/\varepsilon)\boldsymbol{q}\otimes\boldsymbol{a} & -\boldsymbol{q}^{\times} \end{pmatrix}. \tag{42}$$

Let the wave which propagates a stratified isotropic medium be polarized in the plane of incidence: $\boldsymbol{H}_\tau(z) = H_\tau(z)\boldsymbol{b}_0$, $[\boldsymbol{q}\boldsymbol{E}](z) = E_\tau(z)\boldsymbol{b}_0$, where the unit vector $\boldsymbol{b}_0 = \boldsymbol{b}/\sqrt{\boldsymbol{b}^2}$ is co-directed with $\boldsymbol{b}$, and H_τ and E_τ are scalar tangential components of $\boldsymbol{H}$ and $\boldsymbol{E}$, respectively. Then for the profile $\varepsilon(z) = a + b/z^2$, $a = \boldsymbol{b}^2$ from the system (38) with the matrix $\mathbb{M}(z)$ (42), the following two scalar equations for H_τ and E_τ can be found

$$\frac{\mathrm{d}H_\tau}{\mathrm{d}z} = \mathrm{i}k\frac{b}{z^2}E_\tau \qquad \frac{\mathrm{d}E_\tau}{\mathrm{d}z} = \mathrm{i}kH_\tau. \tag{43}$$

Their solution is

$$\begin{aligned} H_\tau &= C_1(z/z_0)^{-\gamma} + C_2(z/z_0)^{\gamma-1} \\ E_\tau &= \frac{z_0}{\mathrm{i}kb}\left[-C_1\gamma(z/z_0)^{-(\gamma-1)} + C_2(\gamma-1)(z/z_0)^{\gamma}\right] \end{aligned} \tag{44}$$

where C_1 and C_2 are constants of integration, $\gamma = \frac{1}{2}(1-\sqrt{1-4k^2b})$ is the parameter introduced earlier. From (40) with the use of $\mathbb{V}(z)$ (42), we determine the vector $\boldsymbol{H}$ of the magnetic field intensity

$$\boldsymbol{H}(z) = H_\tau\boldsymbol{b}_0 - \sqrt{\boldsymbol{b}^2}E_\tau\boldsymbol{q}$$

or

$$\begin{aligned} \boldsymbol{H}(z) = {} & \left[C_1(z/z_0)^{-\gamma} + C_2(z/z_0)^{\gamma-1}\right]\boldsymbol{b}_0 \\ & -\frac{z_0\sqrt{a}}{\mathrm{i}kb}\left[-C_1\gamma(z/z_0)^{-(\gamma-1)} + C_2(\gamma-1)(z/z_0)^{\gamma}\right]\boldsymbol{q}. \end{aligned} \tag{45}$$

The constants C_1 and C_2 in (45) depend on the initial vector $\boldsymbol{H}(z_0)$ taken at the reference point $z = z_0$. If $\boldsymbol{H}(z_0) \equiv \boldsymbol{H}^{(1)}(z_0) = \boldsymbol{b}_0$ then $C_1 = (\gamma-1)/(2\gamma-1)$, $C_2 = \gamma/(2\gamma-1)$ and

$$\begin{aligned} \boldsymbol{H}^{(1)}(z) = {} & \frac{1}{2\gamma-1}\left[(\gamma-1)(z/z_0)^{-\gamma} + \gamma(z/z_0)^{\gamma-1}\right]\boldsymbol{b}_0 \\ & -\frac{z_0\sqrt{a}\gamma(\gamma-1)}{\mathrm{i}kb(2\gamma-1)}\left[-(z/z_0)^{-(\gamma-1)} + (z/z_0)^{\gamma}\right]\boldsymbol{q}. \end{aligned} \tag{46}$$

The second case is $\boldsymbol{H}(z_0) \equiv \boldsymbol{H}^{(2)}(z_0) = \boldsymbol{q}$ with $C_1 = -C_2 = -\mathrm{i}kb/\left[(2\gamma - 1)\sqrt{a}z_0\right]$ and

$$\boldsymbol{H}^{(2)}(z) = \frac{\mathrm{i}kb}{z_0\sqrt{a}(2\gamma - 1)}\left[-(z/z_0)^{\gamma-1} + (z/z_0)^{-\gamma}\right]\boldsymbol{b}_0 + \frac{1}{2\gamma - 1}\left[\gamma(z/z_0)^{-(\gamma-1)} + (\gamma - 1)(z/z_0)^{\gamma}\right]\boldsymbol{q}. \tag{47}$$

Any arbitrary initial vector $\boldsymbol{H}(z_0)$ situated in the $(\boldsymbol{b}, \boldsymbol{q})$-plane can be decomposed as $\boldsymbol{H}(z_0) = k_1\boldsymbol{b}_0 + k_2\boldsymbol{q}$ and in view of the linearity of the basic equations the dependence $\boldsymbol{H}(z)$ on the coordinate z in this case is given by $\boldsymbol{H}(z) = k_1\boldsymbol{H}^{(1)}(z) + k_2\boldsymbol{H}^{(2)}(z)$, where $\boldsymbol{H}^{(1)}(z)$ and $\boldsymbol{H}^{(2)}(z)$ are determined using formulae (46) and (47), respectively. On the other hand, we can represent this dependence of $\boldsymbol{H}(z)$ in the evolutional form $\boldsymbol{H}(z) = \Omega_{z_0}^{z}\boldsymbol{H}(z_0) = \Omega_{z_0}^{z}(k_1\boldsymbol{b}_0 + k_2\boldsymbol{q})$, where $\Omega_{z_0}^{z}$ is the exact evolution operator. It is not difficult to see that the evolution operator $\Omega_{z_0}^{z}$ constructed in this way with the use of (46) and (47) coincides completely with the geometro-optical operator (37) and thus the zeroth-order approximation of tensor geometrical optics already leads to an exact solution for the profile under consideration. Further investigation is needed to establish the first correction given by formula (13). It may well be that this correction is not equal to zero.

5. Conclusion

Although they are analogues of the corresponding scalar quantities, the tensor eikonal and the normal refraction tensor contain considerably more information. This information concerns not only the material parameters of the inhomogeneous anisotropic medium but also polarization characteristics of the eigenwaves in such a medium. In making the transition to the generalized tensor quantities we encounter the problem of non-commutation of the normal refraction tensors N, and as a result disentangling the evolution operator (propagator) is needed. Like matrix algebras, the algebras of the normal refraction tensors are non-Abelian. The example of section 4 points to the complex connection between the fundamental characteristics of photons such as normal, ray, spin, wave and geometro-optical solutions for stratified media. The formulae obtained are well correlated to the conclusions and critical notes expressed in [13] in connection with the gauge of radiation and quantization of the electromagnetic field. The method proposed here enables not only an approximate calculation of the evolution operators, but gives a method for transition from the primal problem to its inverse.

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